

Correlations in Networks associated to Preferential Growth

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Combinations of random and preferential growth for both on-growing and stationary networks are studied and a hierarchical topology is observed. Thus for real world scale-free networks which do not exhibit hierarchical features preferential growth is probably not the main ingredient in the growth process. An example of such real world networks includes the protein-protein interaction network in yeast, which exhibits pronounced anti-hierarchical features.

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One feature that many complex networks show is scale-free degree distribution of vertices, that is the probability of finding a vertex of degree k follows $P(k) \propto k^{-\gamma}$. A popular explanation for the scale-free degree distribution of vertices is preferential attachment [1, 2] in which new vertices tend to connect themselves to already highly connected vertices. In addition to the degree distribution there are additional topological measures that can be used to characterize networks, for example degree-degree correlations, that is “who is connected to who?”. An understanding of by what process networks emerge should then include an understanding of the corresponding topological measures both for real networks and for networks models [3, 4, 5, 6]. In particular it has been observed that protein-protein networks have quite different degree-degree correlations than the Internet [7], although both molecular networks and the Internet show scale-free features. In the present paper we investigate versions of preferential attachment both for on-growing and stationary networks, and study the degree distribution and the degree-degree-correlations. Our conclusion is that preferential attachment is robust with respect to a hierarchical type of degree-degree correlations. As a consequence, real networks which do not have this type of degree-degree correlations are unlikely to have evolved by a version of preferential attachment.

A network, or more formally a graph, $G(V, E)$ consists of a set of vertices V and a set of edges E which connect pairs of vertices in the network. It can both be ordered and unordered pairs depending if the network is directed or not. We only consider undirected networks here. When generating such a network we consider four elementary processes: addition or removal of respectively vertices or edges. Here we use preferential attachment when adding new vertices or edges to the graph, either preferential attachment in itself or combined with random attachment. We will furthermore consider both a

growing network, and a non growing network evolving by addition and removal of vertices and edges at steady state conditions.

I. GROWING NETWORKS

First let us consider a network grown to some number of vertices N , that we fix from the beginning (typically we use $N = 10^3$). The network grows to this size by a process where we at each step do the following:

- With probability p a new vertex is added and connected with an edge to a preferentially selected vertex.
- With probability $1 - p$ a new edge is added between two vertices which are
 - with probability q both chosen preferentially.
 - with probability $1 - q$ one vertex is chosen preferentially and the other vertex is randomly chosen.

Double edges or loops are not allowed and therefore each time we add an edge to the network, a check is performed. If the connection is not valid, one attempts to put the edge somewhere else. This will always be possible, except for some non important cases where the network is very small. To have good statistics a number of networks are grown to the desired size N by the rules above. Also for every network that is produced one makes a sample of randomized networks with exactly the same degree distribution as the grown network, as described in [3]. We look at the like-hood of having a connection between vertices of edge-degree K_1 to vertices of edge-degree K_2 in the real network and compare it with the probability of finding the same connection obtained in the random sample of network [3]:

$$R(K_1, K_2) = \frac{P(K_1, K_2)}{P_{\text{random}}(K_1, K_2)} \quad (1)$$

The reason for comparing with a set of rewired networks is because of the inherently complicated nature of a network. So far the analytical approaches only applies to

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networks where multiple edges between two vertices and loops are allowed. With a specific degree distribution and not allowing for loops or multiple edges between vertices there is a limited freedom when attaching edges. This restriction will give a preference to small vertices connecting with large vertices in a scale-free network. In order to measure how the correlations in the created network differs from the one expected from a network with the same degree sequence we divide the number of specific connections in the studied network with the number of connections in the randomized networks. In principle one could also have obtained information about this “two-point” correlation from the measure of assortative mixing [8], compared with the set of randomized networks, but the full correlation profile contains more specific details about the topology.

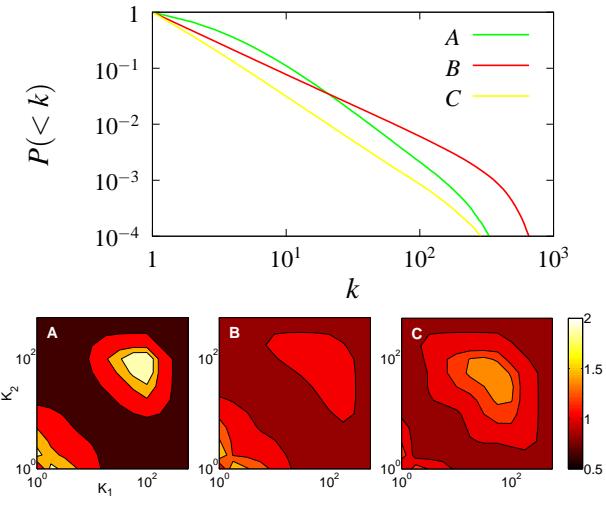


FIG. 1: Growing networks. Cumulative plot of the degree distributions $P(<k)$ for the networks on top and below the correlation profiles $R(K_1, K_2)$ for the respective networks. The number of vertices are 1000 for the correlation profiles and 10000 for the degree distributions. The different networks above are generated with the parameters: (A) $p = 0.4$ and $q = 0$. (B) $p = 0.4$ and $q = 1$. (C) $p = 0.8$ and $q = 1$.

In the figure 1 we show the degree distribution of three differently grown networks labeled A, B and C. That is we consider growth with different rates of edge additions to vertex additions as quantified by p . Further, the given $p = 0.4$ corresponds to adding 4 vertices each with one edge attached preferentially, to every elementary addition of 6 edges. The $q = 1$ and $q = 0$ corresponds to preferential attachment of these 6 edges in both ends, respectively to attachment of one of their ends to a randomly selected vertex. In all cases one obtains scale free networks [9, 10]:

$$P(k) \propto \frac{1}{k^\gamma} \quad (2)$$

with exponent γ that decreases with both p and q . For the three cases in figure 1 we have that (A) $p = 0.4$ with

$q = 0$ gives $\gamma = 2.86$, (B) $p = 0.4$ with $q = 1$ gives $\gamma = 2.14$ and (C) $p = 0.8$ with $q = 1$ gives $\gamma = 2.6$.

Figure 1 (A-C) examines the correlation profile. The overall pattern is that in all cases highly connected vertices tend to connect to highly connected vertices, a feature which in [11] was associated with hierarchical topologies of networks. Also an overall pattern, is that the more edges there are, the more $R(K_1, K_2)$ approaches unity and the hierarchical topologies thus tend to be suppressed by the overall noise. Examining the different types of growth, we furthermore see that the most hierarchical networks are obtained when edges are added randomly in one end and preferentially in the other end. In a somewhat similar vein assortativity was studied in [12].

Since analytical calculations usually are limited to only apply to networks where loops and multiple edges between vertices are allowed, we also investigate what effect this has to the correlation profile by performing the same growth process as before, but with the difference that multiple edges and loops are accepted. Multiple edges and loops are accepted both in the process of growing the networks and the creation of the randomized networks. In figure 2 the correlation profile is visible.

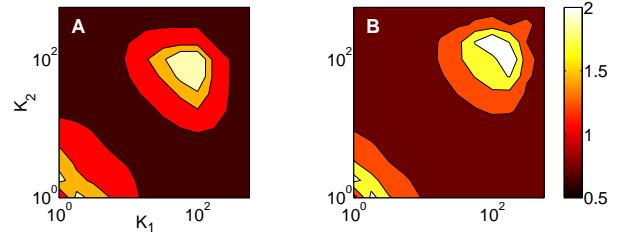


FIG. 2: Comparison of correlation profile for a network process (A) $p = 0.4$ and $q = 0$, and the same process with the only difference that double edges and loops are allowed, (B).

In the figure 2 we see that, even if double edges and loops are allowed, indeed the highly connected vertices are connected more frequently to each other compared with a maximal randomization. However, the peak is shifted towards higher degrees because many edges are allowed between two vertices. If still more edges to vertices are inserted, the differences will be even larger because of the number of double edges and loops that will be created. The preferential attachment is however not the full story of the correlations, there are more to it. In the process of preferential attachment, the oldest vertices tend to become the vertices of highest degree. Furthermore, the insertion of an edge in the network connects two vertices created before the time of the edge insertion. This implicates that when the network is created and all vertex and edge insertions are made, more edges are put between older vertices than the younger vertices simply because the network is smaller in the early stages of the

growth process; thus older vertices have a higher probability to be connected by an edge than the younger vertices created in the later stages. This explains why even if the edges are inserted randomly in both ends one gets a highly hierarchical structure, figure 3A, compared to what is expected from the resulting degree distribution. The degree distribution no longer follows a power law but is still fairly broad. Comparing to the the process where the excess edges are inserted after the insertion of all the vertices to the network, figure 3B, we observe that the hierarchical structure is no longer as apparent as in figure 3A. Furthermore, as a consequence of inserting fewer edges to the older (more connected) vertices the degree distribution is not as broad as in 3A.

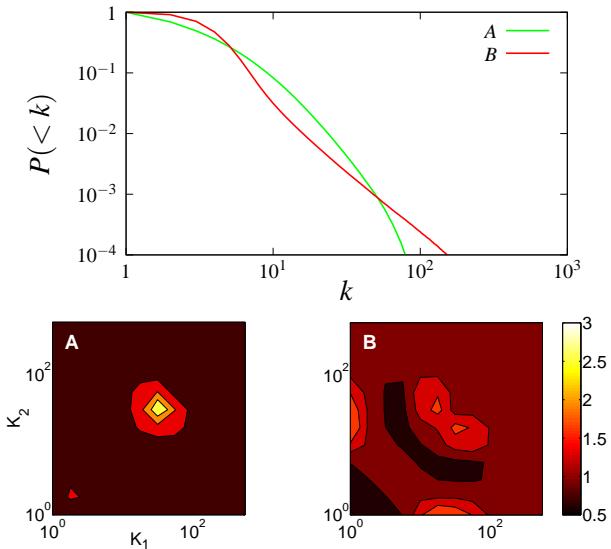


FIG. 3: Cumulative plot of the degree distributions $P(<k)$ for two network processes generating a network of $N = 1000$ vertices. (A) is a process of preferentially vertex and edge insertions, with $p = 0.5$. The edges are inserted randomly in both ends. (B) a process where the excess edges are inserted randomly after all the vertices are inserted preferentially to the network. The number of edges are the same for the two networks, $M = 2000$

II. STATIONARY NETWORKS

Many real world networks are not constantly growing, but may anyway be governed by a growth process, that then should be supplemented by means of elimination of parts of the network. In the case of preferential growth the oldest vertices also become the most central ones which is shown in the degree-degree correlation profile 1. It is therefore of interest to examine what happens if the oldest vertices may be randomly eliminated. This is investigated in the following steady state model for growth and elimination in networks.

Given we want a network consisting of N vertices, we grow the network as before, but in addition add a removal step at any time the number of vertices exceeds N . The total algorithm then reads:

- With probability p a new vertex is added and connected with an edge to a preferentially selected vertex.
- With probability $1 - p$ a new edge is added between two vertices which are
 - with probability q both chosen preferentially.
 - with probability $1 - q$ one vertex is chosen preferentially and the other vertex randomly chosen.
- If $\#\text{vertices} > N$, remove a random vertex n and all vertices that after the removal of n becomes isolated.

At given time-steps (typically at the order of the size of the network), randomizations of the network are made in order to calculate the degree-degree correlation profiles. Figure 4 demonstrates that now, with both growth and elimination, the scale invariance is broken. This is a striking difference to the original Simon [13] model of “rich get richer”. In his model money was assigned to people stochastically with a probability given by their present wealth leading to a power law distribution of wealth.

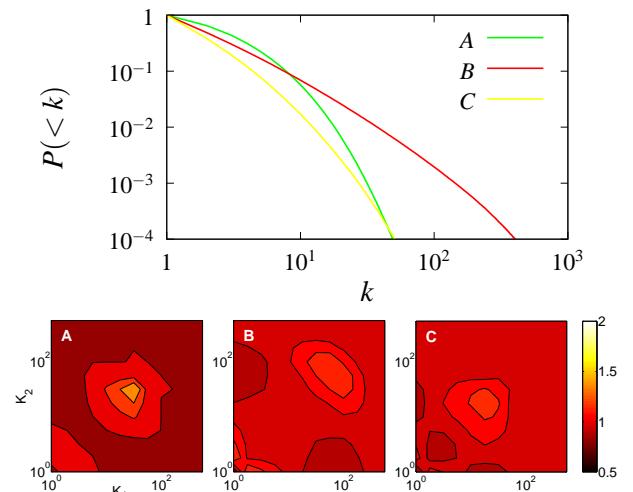


FIG. 4: Stationary networks. Cumulative plot of the degree distributions $P(<k)$ for the networks on top and below the correlation profiles $R(K_1, K_2)$ for the respective networks. The number of vertices are 1000 for the correlation profiles and 10000 for the degree distributions. The different networks above are generated with the parameters: (A) $p = 0.4$ and $q = 0$. (B) $p = 0.4$ and $q = 1$. (C) $p = 0.8$ and $q = 1$.

In his case one also obtains a power law distribution if one randomly eliminates agents independently of their

wealth, see also [14]. The reason for the different behavior in the network case is due to the fact that when one eliminates vertices with few connections, then with high probability one also reduces the number of connections for the vertices with high degrees.

Considering the correlation profiles for the steady state networks one first of all notices that hierarchical features remain. Further, when comparing to the steadily growing networks the hierarchical features are suppressed. Also notice that for the high p or low q the degree distribution became close to exponential, a feature that in itself will diminish the importance of the edge degree as an informative characteristic of the vertex structure. However, the relative strength of observed correlations for steady state networks are qualitatively similar to what was obtained for the growing networks.

In summary we have shown that preferential attachment and continuous edge insertions leads to a rather robust characteristic type of hierarchical degree-degree correlations. Thus for real world scale-free networks that does not exhibit hierarchical features, preferential growth is probably not the main ingredient in forming their topology. An example of such real world networks includes the protein-protein interaction networks in yeast, which exhibits pronounced anti-hierarchical topology [3, 11]. Thus the robustness of the hierarchical topology that preferential attachment gives rise to, points to some difficulty in the preferential attachment scenario put forward in [15] for protein-protein networks, notwithstanding the fact that it was found that the older proteins were observed to be more connected than the younger ones.

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